Time Complexity

Chapters 7

Complexity

- Even when problems are decidable (computable)
  - They may not be feasible
  - Solutions required too much time or memory
- Time Complexity

DEFINITION 7.1

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f : \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

Big-O Notation

- Asymptotic analysis
  - Estimate the run time of an algorithm for large inputs
  - Only consider the highest order terms
    - Disregard coeffiecent of the term and lower order terms

DEFINITION 7.2

Let f and g be functions $f, g\colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = O(g(n))}$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$,

$$f(n) \leq c\, g(n).$$

When $f(n) = O(g(n))$, we say that $g(n)$ is an ***upper bound*** for $f(n)$, or more precisely, that $g(n)$ is an ***asymptotic upper bound*** for $f(n)$, to emphasize that we are suppressing constant factors.

Analyzing Algorithms

- Given A = { 0^k1^k | $k \geq 0$ }
- Low level description of TM M_1 that decides A

M_1 = "On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
3. Scan across the tape, crossing off a single 0 and a single 1.
4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all the 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

- Complexity of this TM will be the number of steps for a given input length
  - Consider some stages separately
    - 1 - Scans for 0*1* form by moving head across tape: n steps = O(n)
    - 2 and 3 – a scan for every pair of 0's and 1's: At most n/2 scans
      - Each scan is O(n) steps: $n/2 * O(n) = O(n^2)$
    - 4 – Single scan to find remaining inputs: n steps = O(n)
  - $O(n) + O(n^2) + O(n) = O(n^2)$

Time Complexity Class

DEFINITION 7.7

Let $t\colon \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, $\mathbf{TIME}(t(n))$, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

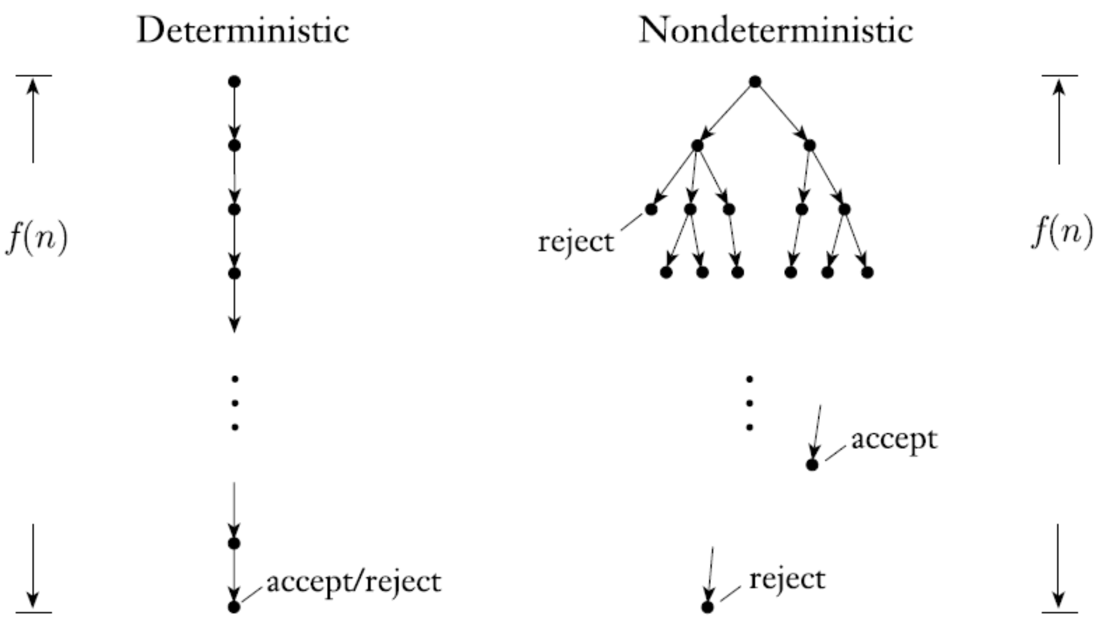
- From previous example
  - $A \in \mathrm{TIME}(n^2)$ because M_1 decides A in time $O(n^2)$

TM Variants Affect Time Complexity

- Given a an O(t(n)) multitape TM
  - The equivalent single tape will run in $O(t^2(n))$

DEFINITION 7.9

Let N be a nondeterministic Turing machine that is a decider. The ***running time*** of N is the function $f\colon \mathcal{N}\longrightarrow\mathcal{N}$, where $f(n)$ is the maximum number of steps that N uses on any branch of its computation on any input of length n, as shown in the following figure.



Section 7.2 Class P

- Problems solved in polynomial time, $O(n^c)$
  - Generally considered to be small and easily computable for large inputs

- Polynomially equivalent
  - Two computational models are Polynomially equivalent if simulating one with the other only increases by polynomial time.
    - Ex. Multitape to singletape TM

DEFINITION 7.12

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

- Important because
  - P is invariant for all models of computation
  - P roughly corresponds to the class of problems that are realistically solvable on a computer

Section 7.3 Class NP

- Not every problem is solvable in polynomial time
- We may be unsure if a language is class P
  - If we can acquire a solution we can try to verify it in polynomial time
  - Easier to verify, harder to determine existence

DEFINITION 7.18

A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

- c = certificate or proof
  - Additional information used to show member ship in language A

NP Class

- Nondeterministic polynomial time
  - class of languages that have polynomial time verifiers
  - Solvable by using a nondeterministic TM in polynomial time

- class P (all problems solvable, deterministically, in polynomial time) is contained in NP (problems where solutions can be verified in polynomial time)
  - P = class of languages for which membership can be decided quickly
  - NP = class of languages for which membership can be verified quickly

DEFINITION 7.21

$$\text{NTIME}(t(n)) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time nondeterministic Turing machine}\}.$$