Mathematical Review

Functions

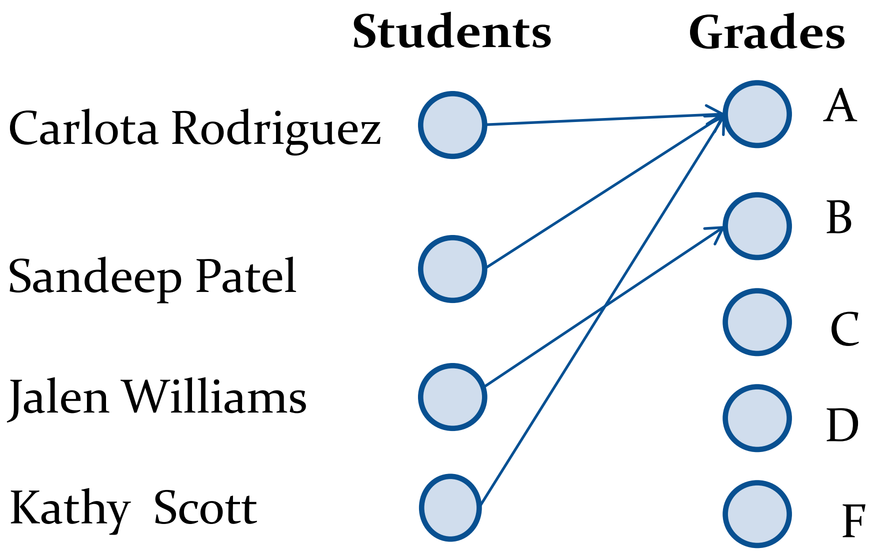
Functions

Functions

- **Definition**: Let A and B be nonempty sets.
  - A ***function*** f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B.

- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

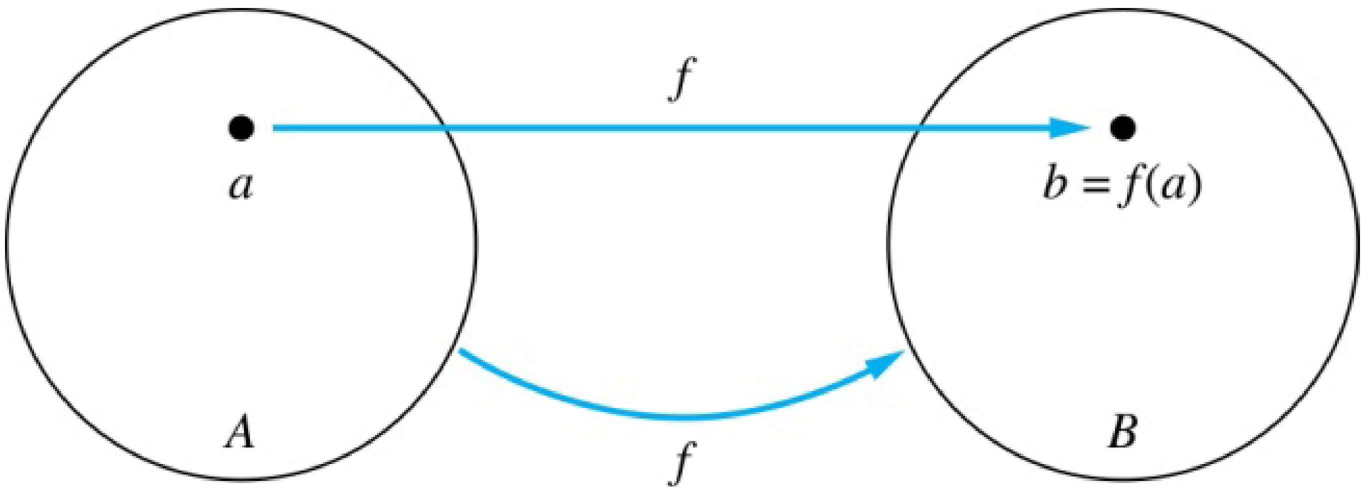
- Functions are sometimes called **mappings** or **transformations**.



Functions

Given a function $f: A \to B$:

- We say f *maps* A to B or
  f is a *mapping* from A to B.
- A is called the ***domain*** of f.
- B is called the ***codomain*** of f.



- If $f(a) = b$,
  - then b is called the ***image*** of a under f.
  - a is called the ***preimage*** of b.

Functions

- The **range** of f is the set of all images of points in A under f.
  - We denote it by f(A).

- Two functions are equal when
  1. they have the same domain
  2. the same codomain
  3. map each element of the domain to the same element of the codomain

Representing Functions

- Functions may be specified in different ways:

  - An explicit statement of the assignment.

    Students and grades example.

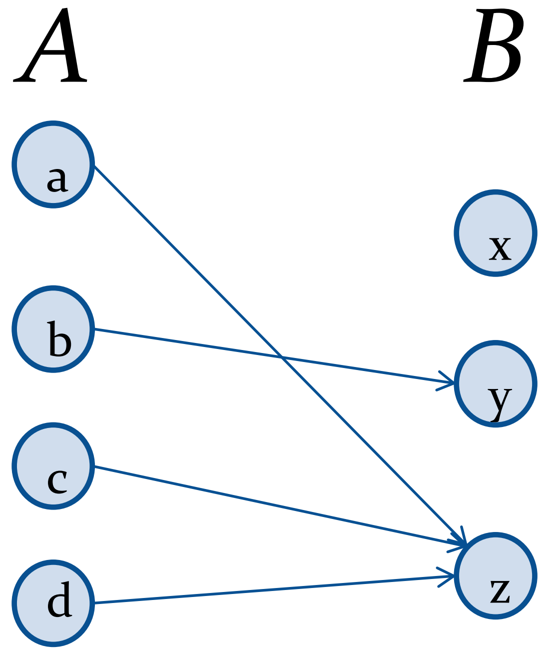
  - A formula.

    $f(x) = x + 1$

  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number

Questions

f(a) = ?	z
The image of d is ?	z
The domain of f is ?	A
The codomain of f is ?	B
The preimage of y is ?	b
$f(A)$ = ?	{y,z}
The preimage(s) of z is (are) ?	{a,c,d}

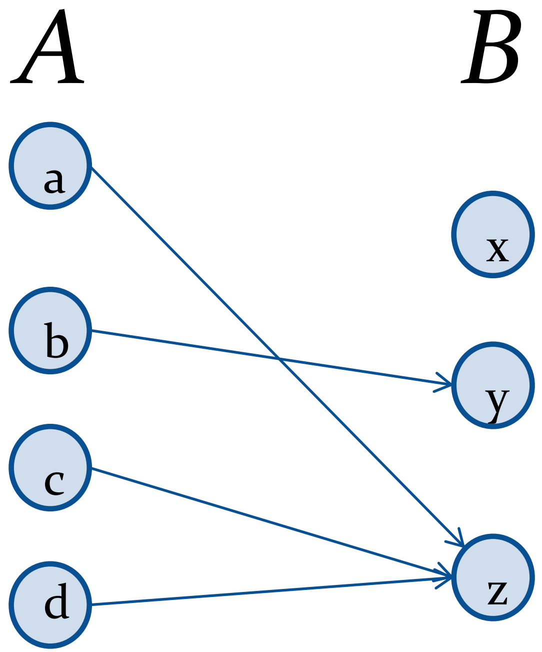


Question on Functions and Sets

- If $f : A \rightarrow B$, then

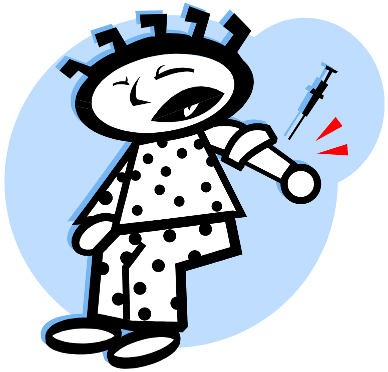
$f\{a,b,c,d\}$ is ? $\{y,z\}$

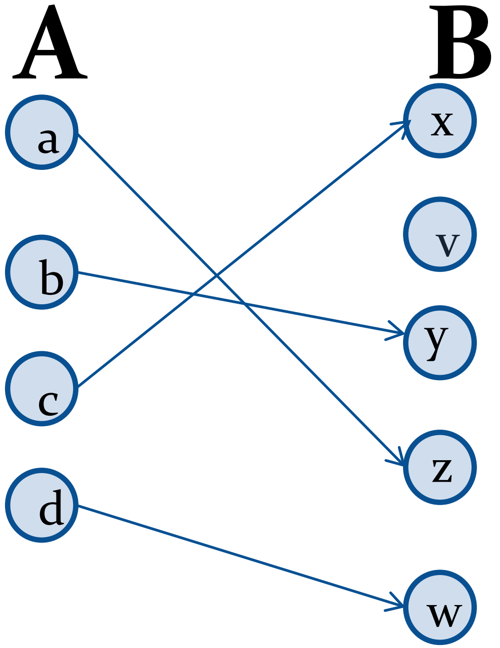
$f\{c,d\}$ is ? $\{z\}$



Injections

- **Definition**: A function f is said to be ***one-to-one***, or ***injective***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
  - A function is said to be an ***injection*** if it is a one-to-one mapping.

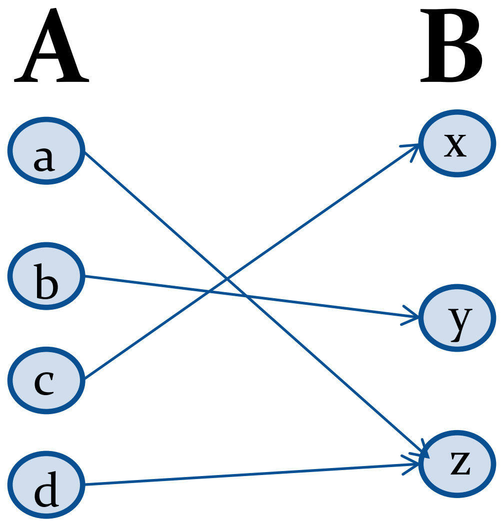




Surjections

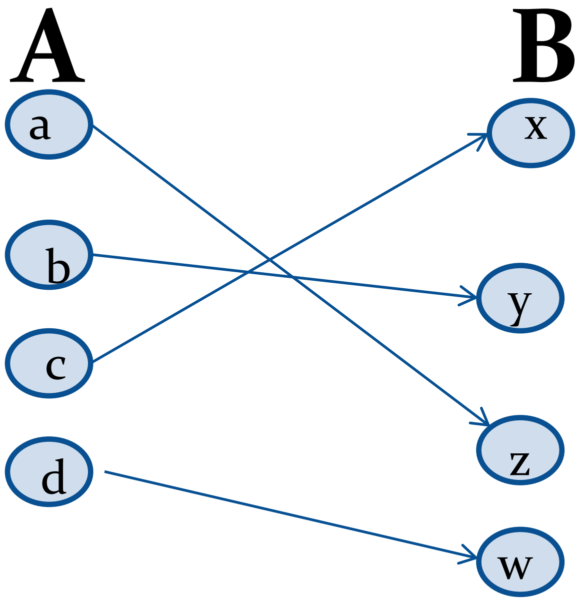
- **Definition**: A function *f* from *A* to *B* is called ***onto*** or ***surjective***, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$

  - A function *f* is called a ***surjection*** if it is *onto*.

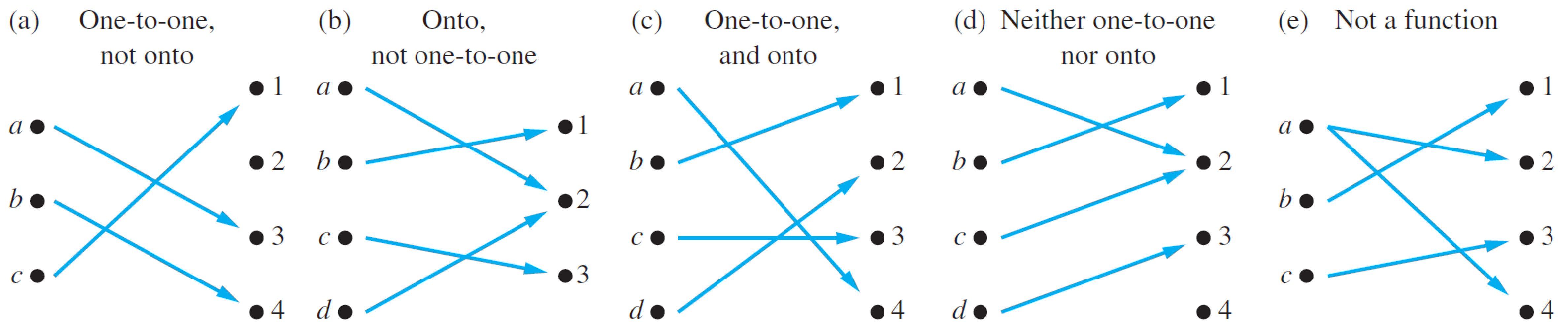


Bijections

Definition: A function f is a ***one-to-one*** *correspondence*, or a ***bijection***, if it is both one-to-one and onto (surjective and injective).



Examples of Different Correspondences



Showing that f is one-to-one or onto

- **Example 1**: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$.
  - Is f an onto function?

- **Solution**: Yes, f is onto since all three elements of the codomain are images of elements in the domain.
  - If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

- **Example 2**: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

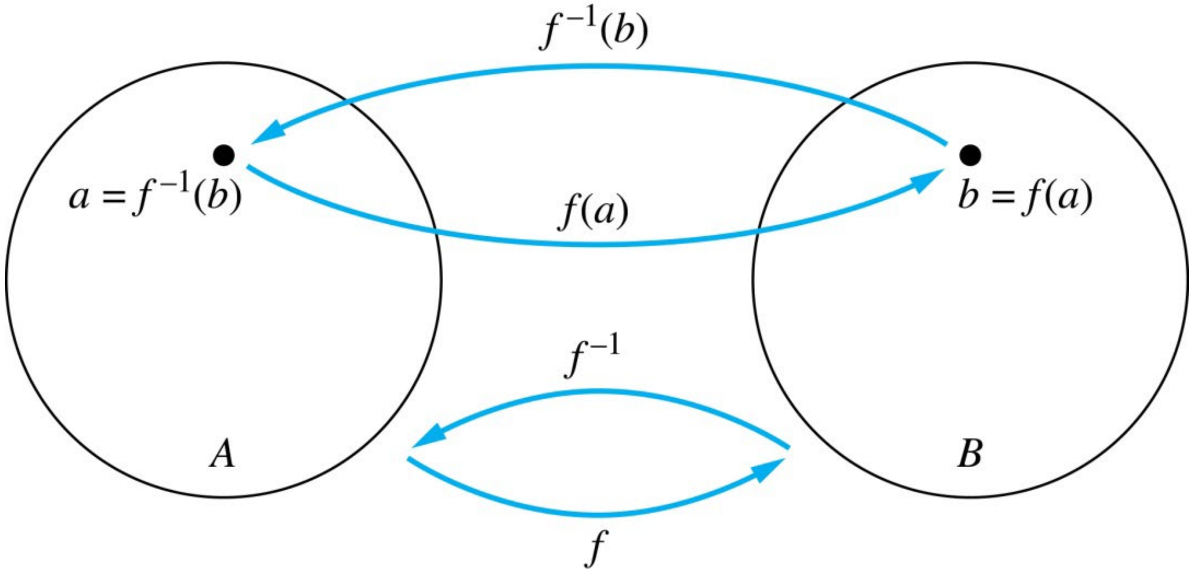
- **Solution**: No, f is not onto because there is no integer x with $x^2 = -1$ (negative integers), for example.

Inverse Functions

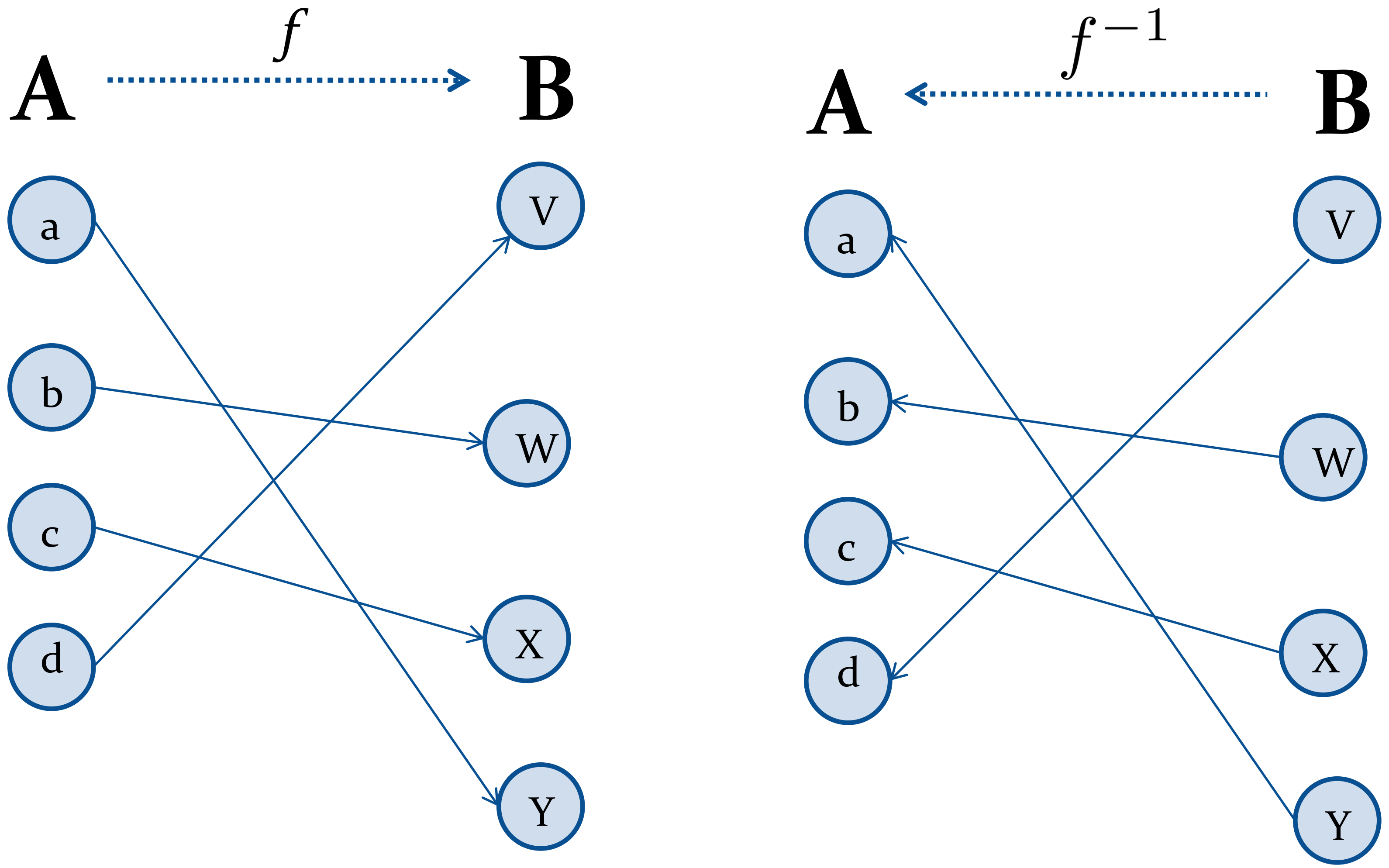
- **Definition**: Let f be a bijection from A to B.
  - Then the ***inverse*** of f, denoted f^{-1}, is the function from B to A defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

  - No inverse exists unless f is a bijection.



Inverse Functions



Questions

- **Example 1**: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.
  - Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one and onto correspondence.

The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

- **Example 2**: Let f: **Z** → **Z** be such that $f(x) = x + 1$.
  - Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one and onto correspondence.

The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

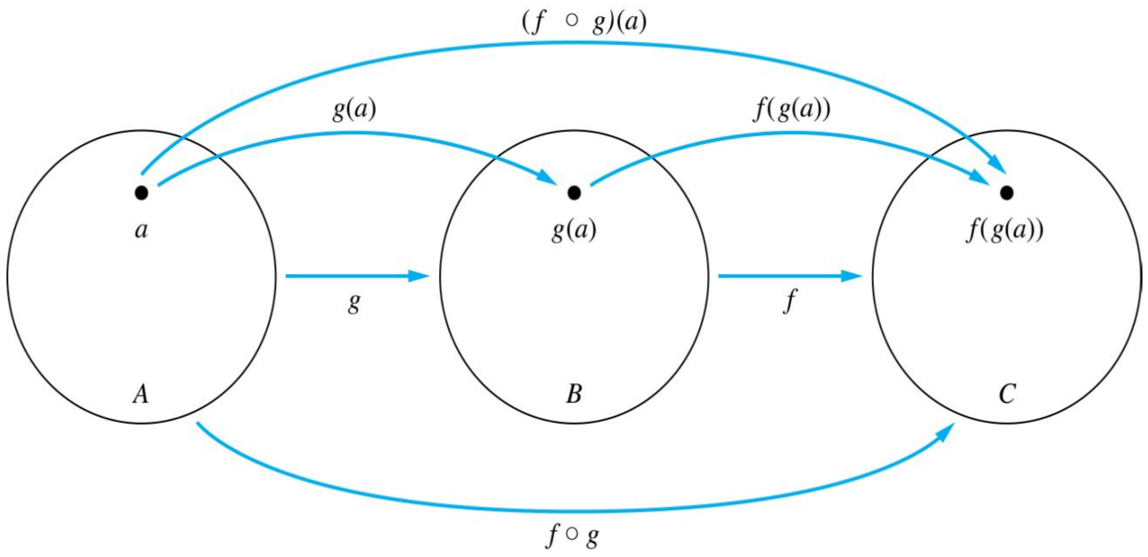
- **Example 3**: Let *f*: **R** → **R** be such that $f(x) = x^2$.
  - Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one .

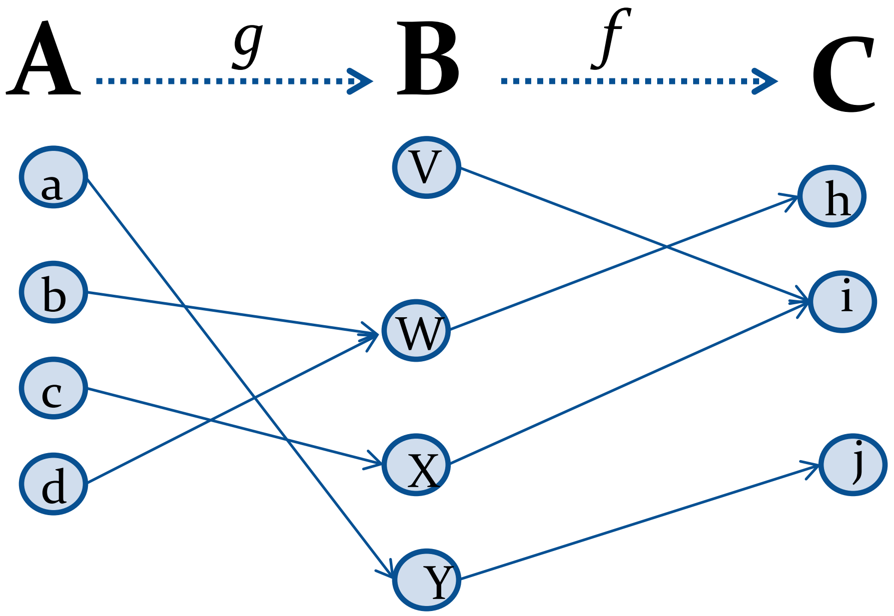
Composition

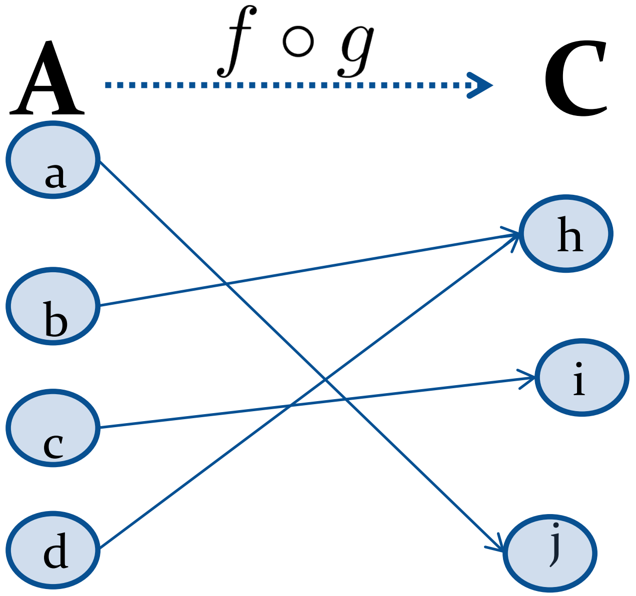
- **Definition**: Let two functions f: $B \rightarrow C$, g: $A \rightarrow B$.
  - The ***composition*** *of f with g*, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$



Composition





Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

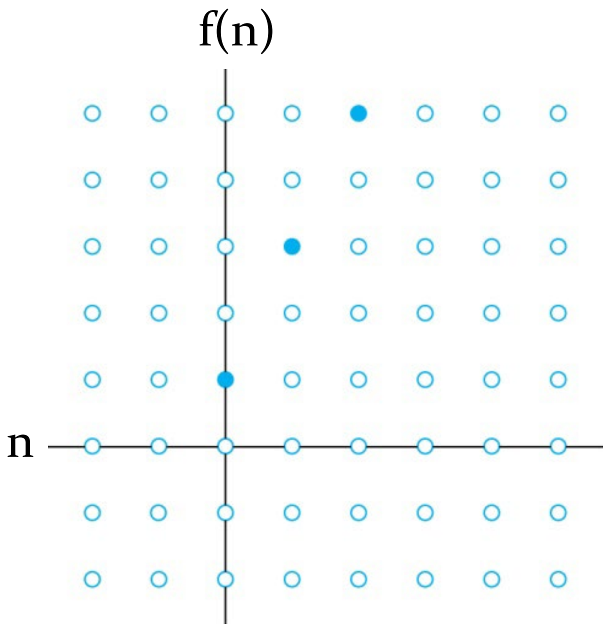
$$f(g(x)) = (2x+1)^2$$

and

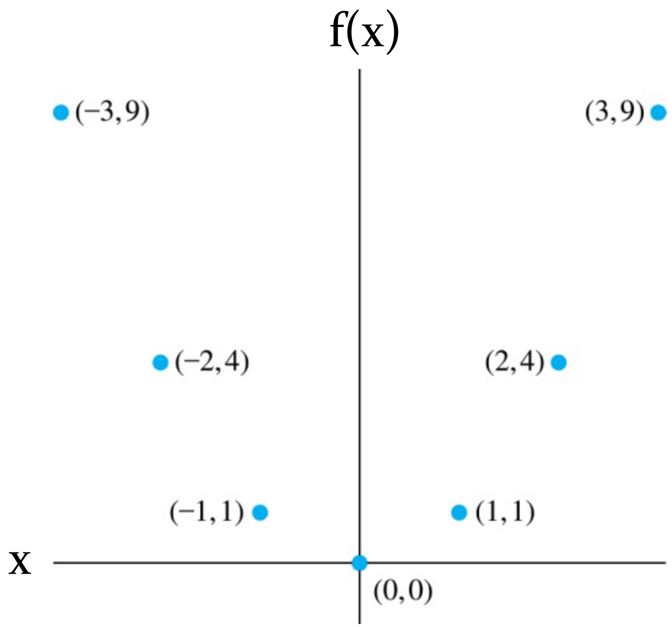
$$g(f(x)) = 2x^2 + 1$$

Graphs of Functions

- Let f be a function from the set A to the set B.
  - The ***graph*** of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z